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A Nonaveraging Set of Integers With a Large Sum of Reciprocals

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Abstract. A set of integers is constructed with no three elements in arithmetic progression and with a rather large sum of reciprocals.

It is a famous open question due to Erdös [2] whether every infinite sequence of positive integers a_i $(i = 1, 2, \dots)$ such that

$$\sum_{i=1}^{\infty} \frac{1}{a_i} = \infty$$

contains arbitrarily long arithmetic progessions. It is not even known whether there exist sequences a_i containing no three terms in arithmetic progression (called for the sake of brevity nonaveraging sets) such that the sum $\sum_{i=1}^{\infty} 1/a_i$ is arbitrarily large.

Gerver (see [4]) constructed sequences containing no k-term arithmetic progression with the sum of reciprocals greater than $(1 - \varepsilon)k \cdot \log k$, where any $\varepsilon > 0$ is appropriate for all but a finite number of integers $k \geqslant 3$.

A well known nonaveraging set apparently first studied by G. Szekeres (see [3]), consists of the numbers $1 + 3^{\alpha_1} + 3^{\alpha_2} + \cdots + 3^{\alpha_k}$, where $k \geqslant 0$ and $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_k$. Denoting it by S we have

$$3.00793 < \sum_{a \in S} \frac{1}{a} < 3.00794$$

(cf. [5] where the value of the sum is given as 3.007).

The aim of this note is to construct a nonaveraging set of integers with the sum of reciprocals appreciably greater than $\sum_{a \in S} 1/a$. The construction uses the idea of Behrend [1]. Let, for $p, q > 0$, $r \geqslant 0$, $B(p, q, r)$ be the set of all integers of the form

$$\sum_{i=1}^{q} k_i (2p - 1)^{i-1},$$

where $0 \leqslant k_i < p$ for $i = 1, 2, \dots, q$ and $\sum_{i=1}^{q} T_p(k_i) = r$ with

$$T_p(k) = \frac{(k - [(p+1)/2]) \cdot (k - [(p-1)/2])}{2}.$$

LEMMA 1. *The set $B(p, q, r)$ is nonaveraging. Moreover, for $s \in B(p, q, r)$ we have $0 \leqslant s < \frac{1}{2}(2p - 1)^q$.*

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The proof is similar to that of Behrend [1] whose k_i^2 has been replaced here by $T_p(k_i)$.

Let Z, T be two finite sets of nonnegative integers, and let z, t be the greatest elements of Z, T respectively. Define (Z, T) by the formula

$$(Z,T) = Z \cup T + m + z + 1 \cup T + 3m + 2t + z + 3 \cup T + 3m + 4t + z + 4,$$

where $m = \max(z, t)$ and $T + x = \{a + x: a \in T\}$.

LEMMA 2. *If Z, T are nonaveraging, so is (Z, T).*

The proof is by straightforward verification.

Now we give the construction of our set. Put

$$Z_0 = \{n \in S: n \leqslant 21\,523\,361\},$$
$$Z_1 = (Z_0, B(4,9,5)), \qquad Z_2 = (Z_1, B(4,10,5))$$

and let for $n \geqslant 3$

$$Z_n = (Z_{n-1}, B(6, n+6, r_n)), \quad \text{where } r_n = \begin{cases} \left[\dfrac{4(n+6)}{3}\right] & \text{for } n \neq 5 \\ 15 & \text{for } n = 5. \end{cases}$$

By definition $Z_0 \subset Z_1 \subset Z_2 \subset \cdots$ and Lemmas 1 and 2 imply that the set $Z = \bigcup_{n=0}^{\infty} Z_n$ is nonaveraging.

Computation performed on the computer ODRA 1305 of the Wrocław University shows that $\sum_{a \in Z} 1/a > 3.00849$. Thus we have established

THEOREM. *There exists a nonaveraging set of integers with the sum of reciprocals greater than* 3.00849.

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